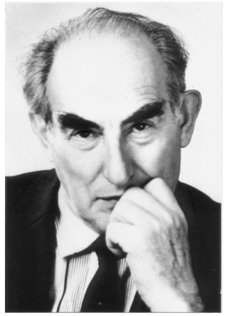
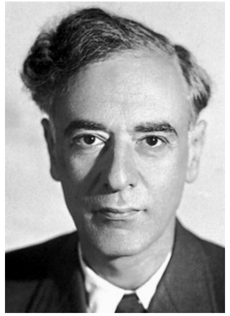
Ginzburg-Landau Theory --- model and characteristic lengths

Discussion the Ginzburg-Landau theory in three parts:

1. Presentation of the model and derivation of the penetration length and coherence length
2. Calculation of the surface energy and categorization of Type I and Type II superconductivity
3. Current-carrying states and phase coherence

GINZBURG – LANDAU THEORY (1950)


Vitaly Ginzburg


Lev Landau

Phenomenological – allows for spatial variations of properties not included in London equations (local electrodynamics)

Order parameter : $\psi(\vec{r})$ complex "wavefunction" $|\psi(\vec{r})|\, e^{i\phi(\vec{r})}$

Superconducting electron density : $|\psi(\vec{r})|^2 = n_s^*$ two-fluid model

Not widely outside Russia in the early days --- thought to be too phenomenological

--- until ---

1959 Gorkov (using thermal Green's functions) showed that


Lev Gorkov

GL $\leftrightarrow$ BCS in limit of near T_c, slowly varying fields and currents and $\psi(\vec{r})$

$\psi(\vec{r}) \sim \Delta(\vec{r})$ energy gap parameter (related to Δ, gap in excitation spectrum)

BCS gap parameter which depends on qp's

Now widely accepted and applicable to many problems

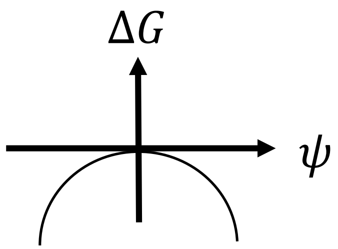
We will follow the presentation (by Ginzburg and Landau) --- not the microscopic derivation (by Gorkov)

GINZBURG – LANDAU MODEL

and free energy near phase transition in $|\psi|^2$ in zero field

$$\Delta G(0) = G_s(0) - G_N(0) = \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4$$

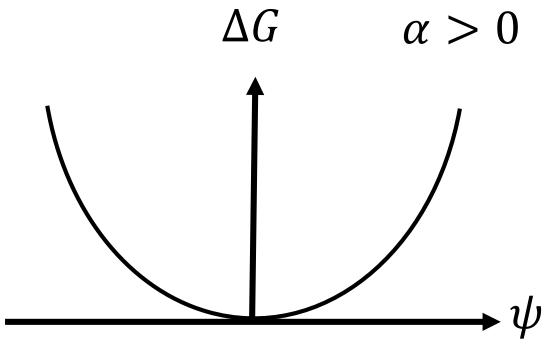
$\beta > 0$ or else minimum G at $|\psi| \longrightarrow \infty$



Simplest possible form:

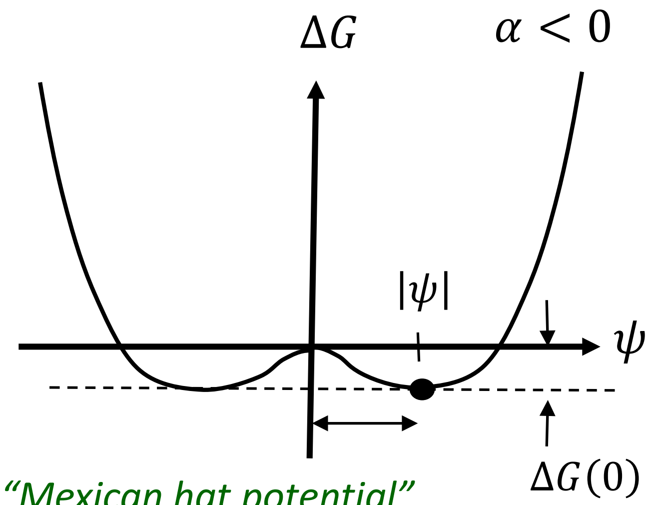
- cannot expand in ψ since G must be real
- odd terms not allowed since G is analytic at $|\psi| = 0$

NORMAL STATE



Minimum ΔG at $|\psi| = 0$

SUPERCONDUCTING STATE



"Mexican hat potential"

Minimum ΔG at $|\psi|^2 = -\frac{\alpha}{\beta} > 0$

$$\left(\frac{d\Delta G}{d|\psi|^2}\right)_T = 0$$ as before (G-C model)

$$\Delta G = -\frac{\alpha^2}{\beta} + \frac{1}{2}\frac{\alpha^2}{\beta} = -\frac{1}{2}\frac{\alpha^2}{\beta} = -\frac{{H_c}^2}{8\pi}$$

"Condensation energy"

$$H_c = \sqrt{\frac{4\pi\alpha^2}{\beta}}$$

$\Delta G < 0 \quad (G_s < G_N)$

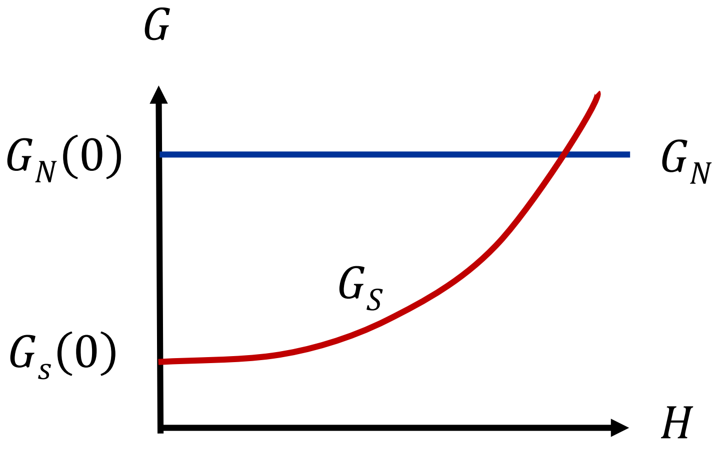
What about α ?

$T > Tc$	NORMAL	$\Delta G = G_S - G_N > 0 \implies \alpha > 0$
$T = T_c$	TRANSITION	$\Delta G = 0 \implies \alpha = 0$
$T < T_c$	SC	$\Delta G < 0 \implies \alpha < 0$

Finite fields $\implies$ allows currents, spatial variations:

MAGNETIC

$$(1)\ Gs(H) = G_s(0) + \underbrace{\frac{H^2}{8\pi}}_{G_{MAG}} \qquad (D = 0)$$



What about β ?

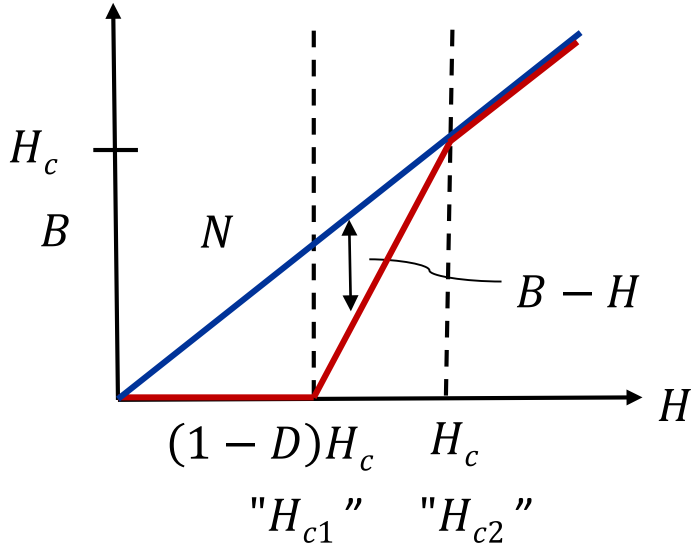
GL assumed $\beta(T) = \beta(T_c) = \text{constant}$

$$H_c^2 = 4\pi \frac{\alpha^2}{\beta} \sim (T_c - T)^2 \implies Hc(T) \sim (T_c - T)$$

which is observed near T_c

With geometry effects (intermediate state):

$$G_s(H) = G_s(0) + \frac{(B-H)^2}{8\pi}$$



If field B penetrates, energy required to expel field is smaller

ELECTRONIC

$$(2)\ G_{EL} = \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi\right|^2 = \quad \text{kinetic energy}$$

$$= \frac{1}{2m^*}\left[\underbrace{\hbar^2\left(\vec{\nabla}|\psi|\right)^2}_{\text{gradients in } |\psi| \text{ (interface N-S)}} + \underbrace{\left(\hbar\vec{\nabla}\phi - \frac{e^*\vec{A}}{c}\right)^2 |\psi|^2}_{\text{kinetic energy of pairs (supercurrent)}}\right] \qquad \text{for} \quad \psi = |\psi|e^{i\phi}$$

gradients in $|\psi|$
(interface N-S)

kinetic energy of pairs (supercurrent)

$$\vec{p} - \frac{e^*\vec{A}}{c} = m^*\vec{v}_s \Longrightarrow$$

$$\therefore n_s{}^*\left(\frac{1}{2}m^*\, v_s{}^2\right)$$

GL only strictly valid at $\sim Tc$ but extensions to lower T are useful and often give useful results

All quantities obtained from 2 experiments λ vs. T and H_c vs. T $\Longrightarrow$ all thermodynamics, electrodynamics, complex cases where fields and currents vary (e.g. intermediate state)

Ginzburg-Landau Theory

Free energy: $\nabla G(\psi, \vec{A}) = G_S - G_N$ $$= \propto |\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi\right|^2 + \frac{1}{8\pi}(B-H)^2$$

CONDENSATION ENERGY

KINETIC ENERGY

MAGNETIC FIELD EXPULSION

$$\frac{\hbar}{2m^*}|\vec{\nabla}(\psi)^2| + ne^*\left(\frac{1}{2}m^* v c^2\right)$$

No fields $\Longrightarrow |\psi| = |\psi_\infty|$

Fields$\Longrightarrow$ minimize total ∇G of sample

Minimize $\int d^3\, x\, \nabla G(\psi, \vec{A})$ wrt ψ and $\vec{A}$

order parameter

magnetic fields

$$\psi \longrightarrow \psi + d\psi$$

$$A \longrightarrow A + dA$$

GL differential equations:

(1) $$\propto \psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)^2 \psi = 0$$

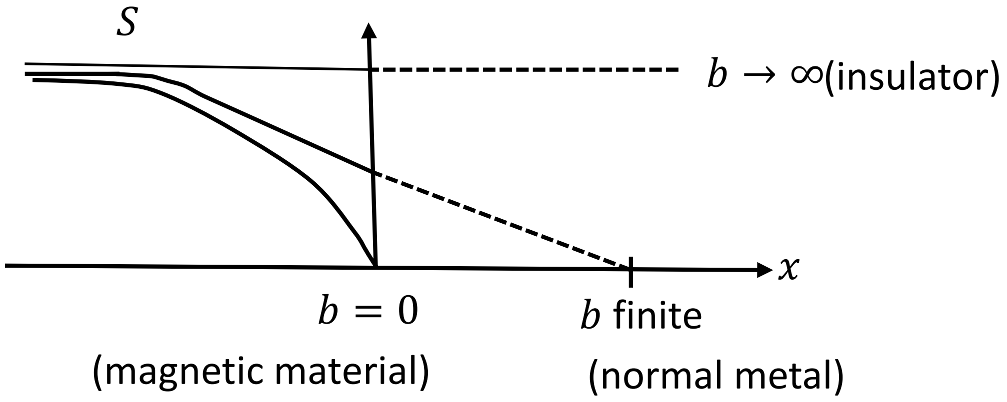
(2) $$\vec{J}_s = \frac{c}{4\pi}\vec{\nabla} \times \vec{B} = \frac{e^*\hbar}{2m^*i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) - \frac{(e^*)^2}{m^*c}|\psi|^2\vec{A}$$

Boundary conditions:

G-L $\quad \left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi \cdot \hat{n} = 0 \quad$ (sufficient to give no current flow)

deGennes $\quad \left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi \cdot \hat{n} = \frac{i}{b}\psi \quad$ (necessary for current-carrying states)

$$\begin{cases} b \to \infty & \text{for insulator} \\ b \text{ finite} & \text{for normal metals,} \\ b \to 0 & \text{for magnetic materials (no SC)} \end{cases}$$



Notes:

(1) LOCAL theory $\vec{J}_s \propto \vec{A}$

(2) EXPANSION in ψ is strictly only valid near T_c but works more widely

(3) SPATIAL theory – allows varying fields and currents

(4) PHENOMENOLOGICAL - $\propto, \beta$ can be determined from measurements of Hc and ns^*(via λ)

$$\vec{A} = 0, \psi \text{ uniform} \Longrightarrow \begin{cases} |\psi|^2 = |\psi_\infty|^2 = {n_s}^* = \dfrac{1}{2} n_s \\ \Delta G = -\dfrac{H^{\ 2}}{8\pi} = -\dfrac{1}{2}\dfrac{\alpha^2}{\beta} \end{cases}$$

ISOLATED SAMPLE – no transport currents

$\psi = \text{real (choose proper gauge)}$

$$(\psi) \quad \frac{\hbar^2}{4m}\nabla^2\psi = \left(\alpha + \frac{e^2}{me^2}A^2\right)\psi + \beta\psi^3$$

$$(A) \quad \nabla^2\vec{A} = \frac{8\pi e^2}{mc^2}\psi^2\,\vec{A} = \frac{4\pi e^2 n_s}{mc^2}\vec{A}$$

using $m^* = 2m, e^* = 2e$ in anticipation of BCS

Length Scales

- Variation of $\vec{A} \Rightarrow$ penetration depth = $\lambda = \left(\frac{mc^2}{4\pi e^2 n_s}\right)^{1/2}$

(regain London for ψ constant) $\psi^2 = n_s{}^* = \frac{1}{2} n_s$ (# pairs)

- Variation $\psi \Rightarrow \xi = GL$ coherence length

Assume no fields $\vec{A} = 0$ Assume no currents ψ real

$$\frac{\hbar^2}{4m} \nabla^2 \psi = \propto \psi + \beta \psi^3$$

$$\frac{\hbar^2}{4m\alpha} \nabla^2 \psi = \psi + \frac{\beta}{\alpha} \psi^3$$

Define $f = \frac{\psi}{|\psi_\infty|} \Rightarrow \psi = |\psi_\infty| f$

$$\frac{\hbar^2}{4m\alpha} \nabla^2 f = f + \frac{\beta}{\alpha} |\psi_\infty|^2 f^3 = f - f^3 \quad \text{using } |\psi_\infty|^2 = -\frac{\alpha}{\beta}$$

$$\frac{\hbar^2}{4m|\alpha|} \nabla^2 f + f - f^3 = 0 \quad \text{since } \alpha < 0$$

Define natural length scale: GL coherence length

$$\boxed{\xi(T) \equiv \left(\frac{\hbar^2}{4m|\alpha|}\right)^{1/2} = \frac{\hbar c}{2\sqrt{2} e H_c(T)\, \lambda(T)}}$$

using $\alpha = -\frac{2e^2}{mc^2} H_c{}^2(T) \lambda^2(T)$

Relation to Pippard coherence length:

No reason why they should be related --- not based on the same physics

$$P: \quad \xi_o = 0.18\,\frac{\hbar v_F}{k_B T_c} \qquad\qquad GL: \quad \xi = \frac{\hbar c}{2\sqrt{2}eH_c(T)\lambda(T)} \sim (1-t)^{-1/2}$$

independent of T $\qquad (1-t) \qquad (1-t)^{-1/2}$ $\qquad$ Diverges at T_c

Connection via BCS theory:

$$\xi(T) = \frac{\pi}{2\sqrt{3}}\left(\frac{H_c(0)}{H_c(T)}\right)\left(\frac{\lambda(0)}{\lambda(T)}\right)\xi_0$$

$$= \frac{0.74\,\xi_o}{(1-t)^{1/2}}$$ clean limit ($\ell \gg \xi_0$) $\quad T{\sim}Tc$

$$= \frac{0.86\,(\xi_o \ell)}{(1-t)^{1/2}}$$ dirty limit ($\ell \ll \xi_0$)$T{\sim}T_c$

$$\xi_o = \sqrt{\frac{3}{2}}\;\frac{\hbar c}{e\pi Hc(0)\lambda(0)}$$

Note: dependence on ℓ also different

$$P: \quad \xi(\ell) \sim \ell \qquad\qquad GL: \quad \xi(\ell) \sim \ell^{1/2}$$

Physical meaning:

Pippard ξ: range of non-local averaging

Ginzburg-Landau ξ: range of order parameter variations

$$\xi^2 \nabla^2 f + f - f^3 = 0$$

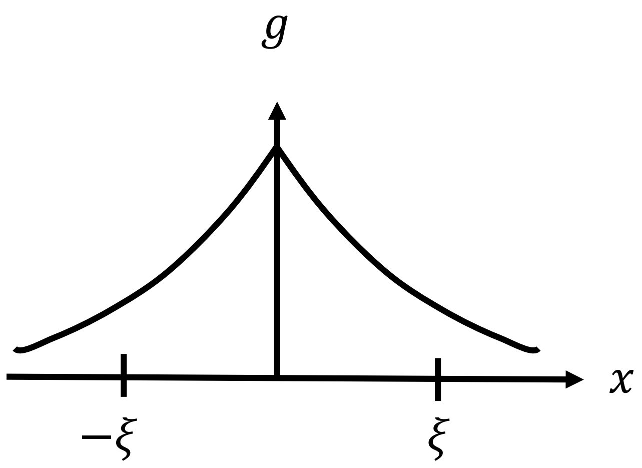
Let $\mathrm{f} = 1 + g$ (perturbation to the bulk order parameter)

$$\xi^2 \nabla^2 g + (1+g) - (1+g)^3 = 0$$

$$\xi^2 \nabla^2 g - 2g + \mathcal{O}(g^2) = 0$$

$$\nabla^2 g \sim \frac{2}{\xi^2}\, g$$

$$g(x) \sim e^{\pm x/(\xi/\sqrt{2})}$$



Sets range for decay of order parameter (cubic term vital)

Scale over which ψ can change with out large cost in free energy

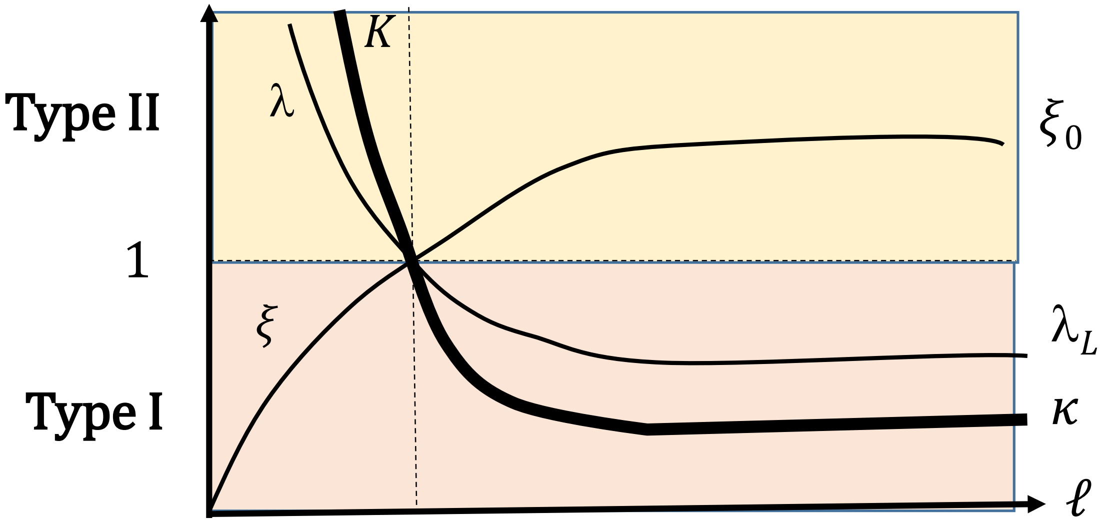
Define GL parameter: $\kappa(T) = \dfrac{\lambda(T)}{\xi(T)}$ ratio of 2 lengths

T dependence:

$$\lambda(T) \sim (1-t)^{-1/2}$$

$$\xi\,(T) \sim (1-t)^{-1/2}$$

$$\kappa(T) \sim \text{constant}$$



ℓ dependence:

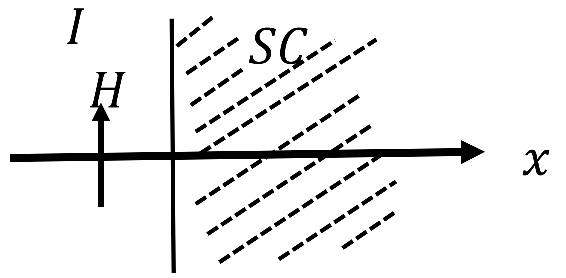
CLEAN	DIRTY
$\lambda \sim \lambda_L, \lambda_L \left(\dfrac{\xi_0}{\lambda_L}\right)^{1/3}$	$\lambda \sim \lambda_L \left(\dfrac{\xi_0}{\ell}\right)^{1/2}$
$\xi \sim \xi_0$	$\xi \sim (\xi_0 \ell)^{1/2}$
$\kappa \sim \dfrac{\lambda_L}{\xi_0}$ ~ constant	$\kappa \sim \dfrac{\lambda_L}{\ell}$

Material	T_c(K)	λ(nm)	ξ_0(nm)	κ
Al	1.2	50	1600	0.03
Sn	3.4	50	230	0.22
Pb	7.0	40	83	0.48
Nb	9.2	85	38	2.24
PbBi	8.3	200	20	10
NbTi	9.5	300	4	75
Nb_3Sn	18	65	3	22
YBCO (a,b)	95	140	1.5	~100
YBCO (c)	95	700	0.3	~1000

BCS:

$$\kappa = \begin{cases} 0.96\,\dfrac{\lambda_L(0)}{\xi_0} & \text{clean limit} \\[2ex] 0.72\,\dfrac{\lambda_L(0)}{\ell} & \text{dirty limit} \end{cases}$$

Variation of Order Parameter near a surface $\xi > \lambda$ (Type I)



GL equations couple φ and $\vec{A}$ so expect φ to depend on field penetration

THICK SAMPLE $d \gg \xi > \lambda_c$

Assume $\delta\Psi$ small → neglect terms in $\vec{\nabla}\psi$ and assume $\psi \sim \psi_\infty$ to get an estimate for $\vec{A}(x)$

GL2: $$\vec{J}_s = \frac{n_s e^2}{mc}\overrightarrow{A}$$

$$\frac{d^2}{dx^2}\vec{A} = -\frac{4\pi nse^2}{me^2}\vec{A} = -\left(\frac{1}{\lambda_L}\right)^2\vec{A}$$

$$A(x) = \lambda_L H e^{-\frac{x}{\lambda_L}}$$

GL1: $$\frac{\hbar^2}{4m}\frac{d^2\Psi}{dx^2} = \left(\propto + \frac{e^2}{mc^2}A^2(x)\right)\Psi(x) + \beta\Psi^3(x)$$

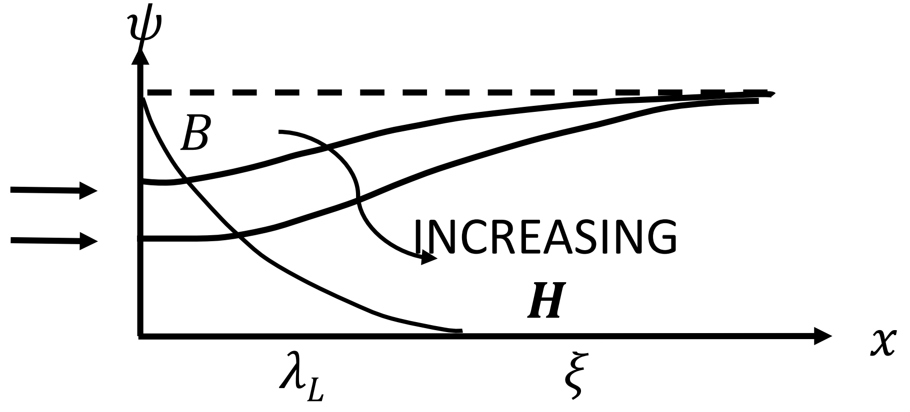
$$\Psi(x) = \Psi_\infty\left\{1 - \frac{\kappa}{2\sqrt{2}(2-\kappa^2)}\left(\frac{H}{H_c}\right)^2\left[e^{-\frac{\sqrt{2}}{\xi}x} - \frac{x}{\sqrt{2}}e^{-\frac{2x}{\lambda}}\right]\right\}$$

B.C. $\vec{A}\cdot\hat{n} = 0 \Rightarrow \frac{d\varphi}{dx} = 0$ at surface

$$\Psi(x) = \psi_\infty\{1 - \frac{\kappa}{2\sqrt{2}(2-\kappa^2)}(\frac{H}{H_c})^2\left[e^{-\frac{\sqrt{2}}{\xi}x} - \frac{x}{\sqrt{2}}\, e^{-\frac{2x}{\lambda}}\right]\}$$

Double exponential in ξ and λ

$\frac{d\psi}{dx} = 0$



$$\frac{\Delta\Psi}{\Psi_0}(x=0) = \frac{\kappa}{4\sqrt{2}}\left(\frac{H}{H_c}\right)^2$$

Suppression of order parameter in field

Full self-consistent solution shows deviation of λ(x) from exponential